

Soliton-like solutions to the ordinary Schroedinger equation^(†)

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Abstract – In recent times it has been paid attention to the fact that (linear) wave equations admit of “soliton-like” solutions, known as Localized Waves or Non-diffracting Waves, which propagate without distortion in one direction. Such Localized Solutions (existing also for K-G or Dirac equations) are a priori suitable, more than gaussian’s, for describing elementary particle motion. In this paper we show that, *mutatis mutandis*, Localized Solutions exist even for the ordinary Schroedinger equation within standard Quantum Mechanics; and we obtain both approximate and exact solutions, also setting forth for them particular examples. In the ideal case such solutions bear infinite energy, as well as plane or spherical waves: we show therefore how to obtain finite-energy solutions. At last, we briefly consider solutions for a particle moving in the presence of a potential.

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1 Introduction

Recently it has been shown —as it had been already realized in old times[1]— that not only nonlinear, but also a large class of linear equations (including, in particular, the wave equations) admit of “soliton-like” solutions. Those solutions[2] are *localized*, and travel along their propagation axis practically without diffracting (at least until a certain field-depth[2,3,4]): Such wavelets were indeed called “undistorted progressing waves” by Courant and Hilbert[1]. Let us recall that their peak-velocity V can assume any values[5,6,2] $0 \leq V \leq \infty$, even if we are mainly interested here in their *localization properties* rather than in their group-velocity. In the case of wave equations, the localized solutions more easy to be constructed in exact form resulted to be the so-called “(superluminal) X-shaped” ones (see Refs.[4,7,8,2], and refs. therein).

The X-shaped waves, long ago predicted[6] to exist within Special Relativity (SR), have been first mathematically constructed[9,2] as solutions to the wave equations in Acoustics[4], and later on in Electromagnetism (namely, to the Maxwell equations[7]), and soon after produced experimentally[10]. Only very recently, *subluminal* localized solutions have been suitably worked out in exact form[11], even for the case of zero speed (“Frozen Waves”).[12]

It was soon thought that, since the mentioned solutions to the wave equations are non-diffractive and particle-like, they may well be related to elementary particles (and to their wave nature)[13,14]. And, in fact, localized solutions have been found for Klein-Gordon and for Dirac equations[13,14].

However, little work[15] has been done, as far as we know, for the (*different*) case of the *Schroedinger* equation*. Indeed, the relation between the energy E and the impulse magnitude $p \equiv |\mathbf{p}|$ is quadratic [$E = p^2/(2m)$] in the non-relativistic case, like in Schroedinger’s, at variance with the relativistic one. But, as we were saying, the nondiffracting solutions, which are essentially superpositions of Bessel beams and are currently called *Localized Waves*, would be quite apt at describing elementary particles: much more than the gaussian waves. In this paper we show that indeed, *mutatis mutandis*, Localized Solutions exist even for the ordinary Schroedinger equation within standard Quantum Mechanics; and we obtain both approximate and exact solutions, also setting

*For some work in connection with the ordinary Schroedinger equation, see for instance, besides [7], also Refs.[14].

forth for them particular examples. In the ideal case such solutions bear infinite energy, as well as spherical or plane waves: we shall therefore show how to obtain finite-energy solutions. At last, we shall briefly consider solutions for a particle moving in the presence of a potential.

Before going on, let us recall that, in the time-independent realm —or, rather, when the dependence on time is only harmonic, i.e., for monochromatic solutions—, the (quantum, non-relativistic) Schroedinger equation is mathematically identical to the (classical, relativistic) Helmholtz equation[16]. And many trains of localized X-shaped pulses have been found, as *superpositions* of solutions to the Helmholtz equation, which propagate, for instance, along cylindrical or co-axial waveguides[17]; but we shall skip all the cases[18] of this type, even if interesting, since we are concerned here with propagation in free space, even when in the presence of an ordinary potential. Let us also mention that, in the *general* time-*dependent* case, that is, in the case of pulses, the Schroedinger and the ordinary wave equation are no longer mathematically identical, since the time derivative results to be of the first order in the former and of the second order in the latter. [It has been shown that, nevertheless, at least in some cases[19], they still share various classes of analogous solutions, differing only in their spreading properties[19]]. Moreover, the Schroedinger equation implies the existence of an *intrinsic* dispersion relation even for free particles.

Another difference, to be kept here in mind, between the wave and the Schroedinger equations is that the solutions to the wave equation suffer only diffraction (and no dispersion) in the vacuum, while those of the Schroedinger equation suffer also (an intrinsic) dispersion even in the vacuum.

Let us repeat that the majority of the ideal localized solutions we are going to construct are endowed with infinite energy. We shall treat also a *finite-energy* case[†] only towards the end of this paper: In fact, infinite-energy solutions themselves, even without truncating them in space and time, results to be rather useful for describing wavepackets in regions not too extended in the transverse direction; as we shall see below.

[†]In such cases the solutions travel undistorted and with a constant speed along a finite depth of field only.

2 Bessel *beams* as localized solutions (LS) to the Schroedinger equation

Let us consider the Schroedinger equation for a free particle (an electron, for example)

$$\nabla^2 \psi + \frac{2im}{\hbar} \frac{\partial \psi}{\partial t} = 0 . \quad (1)$$

If we confine ourselves to solutions of the type

$$\psi(\rho, z, \varphi; t) = F(\rho, z, \varphi) e^{-iEt/\hbar} ,$$

their spatial part F obeys the reduced equation

$$\nabla^2 F + k^2 F = 0 , \quad (2)$$

with $k^2 \equiv p^2/\hbar^2$ and $p^2 = 2mE$ (quantity $p \equiv |\mathbf{p}|$ being the particle momentum, and therefore $k \equiv |\mathbf{k}|$ the total wavenumber). Equation (2) is nothing but the Helmholtz equation, for which various simple localized-beam solutions (LS) are already known: In particular, the so-called Bessel beams[2], which have been experimentally produced since long[20].

Namely, let us now look —as usual— for solutions (cylindrically symmetric with respect to [w.r.t.] the z -axis) of the form

$$\psi(\rho, z; t) = R(\rho) Z(z) T(t) ,$$

and explicitly indicate, mainly for clarity's sake, the subsequent steps. Equation (1) then becomes

$$\left[\frac{1}{\rho R} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial R}{\partial \rho} \right) + \frac{1}{z} \frac{\partial^2 Z}{\partial z^2} \right] = -i \frac{2m}{\hbar} \frac{1}{T} \frac{\partial T}{\partial t} \quad (3)$$

so that

$$- \frac{i2m}{\hbar T} \frac{\partial T}{\partial t} = -k^2 \implies T = e^{-iEt/\hbar} \quad (4)$$

where, let us repeat, $E = p^2/(2m) = k^2\hbar^2/(2m)$ (and in fact the last exponential is often written as $\exp[-i\omega t]$).

Analogously, we have

$$\frac{1}{\rho R} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial R}{\partial \rho} \right) + k^2 = -\frac{1}{z} \frac{\partial^2 z}{\partial z^2} \quad (5)$$

and therefore

$$-\frac{1}{z} \frac{\partial^2 z}{\partial z^2} = k_z^2 \implies z = e^{izk_z}, \quad (6)$$

where the constant $k_z \equiv k_{\parallel} = p_{\parallel}/\hbar \equiv p_z/\hbar$ is the longitudinal wavenumber.[‡] We will suppose $k_z \geq 0$, that is, $p_z \geq 0$, to ensure that we deal with forward traveling beams only.

As a consequence, the (transverse) function $R = R(\rho)$ obeys the equation

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial R}{\partial \rho} \right) + (k^2 - k_z^2) R = 0 \quad (7)$$

which is a Bessel differential equation admitting as solution the Bessel function[§]

$$R(\rho) = J_0(\rho k_{\rho}), \quad (8)$$

where the constant $k_{\rho} \equiv k_{\perp} \equiv p_{\perp}/\hbar$ is the transverse wavenumber, and

$$k_{\rho}^2 = k^2 - k_z^2 \equiv 2mE/\hbar^2 - k_z^2. \quad (9)$$

To avoid any divergencies, it must be $k_{\rho}^2 \geq 0$, that is, $k^2 \geq k_z^2$; namely, it must hold [see (a) in Fig.1] the constraint

$$E \geq \frac{p_z^2}{2m} \quad (10)$$

[Notice that, to avoid the appearance of evanescent waves, one should postulate k_z to be real; but such a condition is already included in our previous assumption that $k_z \geq 0$]. In the following, to simplify the notations, we shall also put $[p \equiv \hbar k]$:

[‡]Since the present formalism is used both in quantum mechanics and in electromagnetism, with a difference in the customary nomenclature, for clarity's sake let us here stress, or repeat, that $k \equiv p/\hbar$; $k_{\rho} \equiv k_{\perp} \equiv p_{\perp}/\hbar$; $\omega \equiv E/\hbar$; while $k_z \equiv k_{\parallel} = p_{\parallel}/\hbar \equiv p_z/\hbar$ is often represented by the (for us) ambiguous symbol β .

[§]The other Bessel functions are not acceptable here, because of their divergence at $\rho = 0$ or for $\rho \rightarrow \infty$.

$$p_\rho \equiv p_\perp .$$

The solution is therefore:

$$\psi(\rho, z; t) = J_0(\rho p_\rho / \hbar) \exp[i(z p_z - Et)/\hbar] \quad (11)$$

together with condition (9). *Equation (11) can be regarded as a Bessel beam solution to the Schroedinger equation.* This result is not surprising, since —once we suppose the whole time variation to be expressed by the function $\exp[i\omega t]$ — both the ordinary wave equation and the Schroedinger equation transform into the Helmholtz equation. Actually, the only difference between the Bessel beam solutions to the wave and to the Schroedinger equation consists in the different relationships among frequency, longitudinal, and transverse wavenumber; in other words (with $E \equiv \omega\hbar$):

$$p_\rho^2 = E^2/c^2 - p_z^2 \quad \text{for the wave equation;} \quad (12a)$$

$$p_\rho^2 = 2mE - p_z^2 \quad \text{for the Schroedinger equation.} \quad (12b)$$

In the case of beams, the experimental production of LSs to the Schroedinger equation can be similar to the one exploited for the LSs to the wave equations (e.g., in Optics, or Acoustics): Cf., e.g. Figure 1.2 in the first one of Refs.[8], and refs. therein, where the simple case of a source consisting in an array of circular slits, or rings, were considered.[¶] In the Table we refer to a Bessel beam of photons, and a Bessel beam of (e.g.) electrons, respectively. We list therein the relevant quantities having a role, e.g., in Electromagnetism, and the corresponding ones for the Schroedinger equation's spatial part $\hbar^2 \nabla^2 F + 2mE F = 0$, with $F = R(\rho) Z(z)$. The second and the fourth lines have been written down for the simple Durnin et al.'s case, when the Bessel beam is produced by an annular slit (illuminated by a plane wave) located in the focus of a lens[20].

[¶]For pulses, however, the generation technique must deviate from Optics', since in the Schroedinger equation case the phase of the Bessel beams produced through an annular slit would depend on the energy.

WAVE EQUATION	SCHROEDINGER EQUATION
$k = \frac{\omega}{c}$	$p = \sqrt{2mE}$
$k_\rho \simeq \frac{r}{f} k$	$p_\rho \simeq \frac{r}{f} p$
$k_\rho^2 = \frac{\omega^2}{c^2} - k_z^2$	$p_\rho^2 = 2mE - p_z^2$
$k_z^2 = \frac{\omega^2}{c^2} (1 - \frac{r^2}{f^2})$	$p_z^2 = 2mE (1 - \frac{r^2}{f^2})$

In this Table, quantity f is the focal distance of the lens (for instance, an ordinary lens in optics; and a magnetic lens in the case of Schroedinger charged wavepackets), and r is the radius of the considered ring. [In connection with the last line of the Table, let us recall that in the wave equation case the phase-velocity ω/k_z is almost independent of the frequency (at least for limited frequency intervals, like in optics), and one gets a constant group-velocity and an easy way to build up X-shaped waves. By contrast, in the Schroedinger case, the phase-velocity of each (monochromatic) Bessel beam depends on the frequency, and this makes it difficult to generate an “X-wave” (i.e., a wave depending on z and t only via the quantity $z - Vt$) by using simple methods, as Durnin et al.’s, based on Bessel beams superposition. In the case of charged particles, one should compensate such a velocity variation by suitably modifying the focal distance f of the Durnin’s lens, e.g. on having recourse to an additional magnetic, or electric, lens.]

Before going on, let us stress that one could easily eliminate the restriction of axial symmetry: In such a case, in fact, solution (11) would become

$$\psi(\rho, z, \varphi; t) = J_n(\rho p_\rho/\hbar) e^{izp_z/\hbar} e^{-iEt/\hbar} e^{in\varphi},$$

with n an integer. The investigation of not cylindrically-symmetric solutions is interesting especially in the case of localized *pulses* (cf. Sect.3): and we shall deal with them below.

3 Localized *pulses* as solutions to the Schroedinger equation (*approximate method*)

Localized (non-dispersive, besides non-diffracting) *pulses* can be constructed, as solutions to the Schroedinger equation, both by having recourse to the standard “paraxial approx-

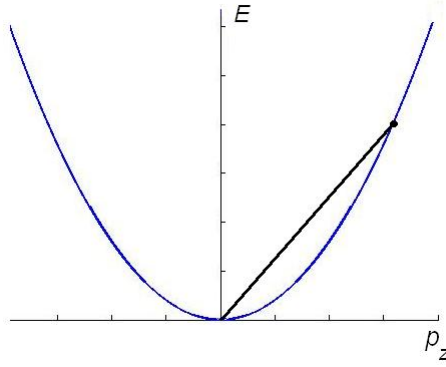


Figure 1: The parabola and the chosen straight-line have equations $E = p_z^2/(2m)$ and $E = Vp_z$, respectively. The intersection of our straight-line with the parabola corresponds to the value $E = 2mV^2$. The allowed region is the one internal to the parabola, since it must be $E \geq p_z^2/(2m)$.

imation”, and in an exact, analytic way. Let us start with the approximate method.

Let us go back, then, to our Bessel beam solution (11), with condition (10). We can obtain localized (non-dispersive) pulses, as solutions to Schroedinger’s equation, by suitably superposing the beam solutions (11), and by selecting in the plane (p_z, E) the straight-line [see Fig.1]:

$$E = Vp_z ; \quad (p_z \geq 0) , \quad (13)$$

with V a chosen constant speed; so that from eq.(10) one gets the important condition

$$E \leq 2mV^2 \quad (14)$$

and eq.(11) can consequently be written

$$\psi(\rho, \zeta) = J_0(\rho p_\rho / \hbar) \exp[ip_z \zeta / \hbar] \quad (11')$$

where now $p_\rho^2 = (2mE - p_z^2) = E(2m - E/V^2)$ and we introduced the new variable

$$\zeta \equiv z - Vt . \quad (15)$$

Localized-wave solutions can be therefore obtained through the superposition (see Fig.1):

$$\Psi(\rho, \zeta) = \mathcal{N} \int_0^{2mV^2} dE J_0 \left(\rho \sqrt{\frac{E}{\hbar^2} \left(2m - \frac{E}{V^2} \right)} \right) \exp \left[i \frac{E}{\hbar V} \zeta \right] S(E) \quad (16)$$

the weight-function $S(E)$ being a suitable energy-spectrum (with the dimensions, as usual, of the inverse of an Energy), while \mathcal{N} is a “normalization” constant which normalizes to 1 the peak-value of $|\Psi|^2$ and (since it multiplies a dimensionless integral) bears the dimensions $[\mathcal{N}] = [L^{3/2}]$, to respect the ordinary meaning of $|\Psi(\rho, \zeta)|^2$. It should be noted that we are integrating, in the space (p_z, E) along the straight-line (13), that is, $E = V p_z$. This corresponds to superposing Bessel beams all endowed with the same phase-velocity $V_{\text{ph}} \equiv V$. The resulting pulse will possess V as its *group-velocity* (namely, as its peak-velocity), since it is well-known that, when the phase-velocity V_{ph} does not depend on the energy or frequency, the resulting pulse happens to travel with the group-velocity $V_g \equiv \partial\omega/\partial k_z = V_{\text{ph}} \equiv V$: cf. refs.[17,2,21] and refs. therein. Due to constraint (14), we are actually integrating along our straight-line from 0 to $2mV^2$ (see Fig.1).

It is important also to note explicitly that each solution $\Psi(\rho, \zeta)$ given by eq.(16), depending on z (and t) only via the variable $\zeta \equiv z - Vt$, does represent a pulse that appear with a constant shape to an observer traveling with speed V along the wave motion-line z : in other words, it represents a pulse which propagates rigidly along z . *Therefore, eqs.(16) are already —as desired— non-dispersing and non-diffracting (“localized”) solutions to the Schroedinger equation.*

Integrals (16), however, appear difficult to be analytically performed, independently of the spectrum $S(E)$ chosen.

To overcome this difficulty, let us rewrite eq.(11') as a function of p_ρ only, by exploiting eq.(12b), which can be written $E^2/V^2 - 2mE + p_\rho^2 = 0$, and yields

$$E = mV^2 \left(1 + \sqrt{1 - \frac{p_\rho^2}{p_{\rho\text{max}}^2}} \right), \quad (17)$$

where

$$p_{\rho\text{max}} = mV,$$

as it comes by deriving eq.(12b) with respect to E .

Therefore, eq.(11') becomes

$$\psi(\rho, \zeta) = J_0(\rho p_\rho / \hbar) \exp \left[i \frac{mV}{\hbar} \zeta \sqrt{1 - \frac{p_\rho^2}{m^2 V^2}} \right] S(p_\rho / \hbar) e^{i \frac{mV}{\hbar} \zeta} \quad (11'')$$

with $0 \leq p_\rho \leq p_{\rho \max}$, where,^{||} let us repeat, $p_{\rho \max} = mV$. Then, the Localized Solutions will be written as

$$\Psi(\rho, \zeta) = \mathcal{N} e^{imV\zeta/\hbar} \int_0^{mV} dp_\rho J_0\left(\frac{\rho p_\rho}{\hbar}\right) S(p_\rho) \exp \left[\frac{imV}{\hbar} \zeta \sqrt{1 - \frac{p_\rho^2}{m^2 V^2}} \right]. \quad (18)$$

Let us notice that, in the new variable p_ρ , the Bessel function, previously written as in eq.(16), gets, as we have seen, the simplified expression $J_0(\rho p_\rho)$.

It is now enough to choose a weight-function S that is strongly bumped around the value p_ρ , in the interval $[0, mV]$, with

$$p_\rho \ll mV, \quad (19)$$

for being able to integrate from 0 to ∞ with a negligible error. Namely, let us now adopt the so-called *paraxial approximation*. Under condition (19), one can approximate the exponential factor as follows:

$$mV \sqrt{1 - \frac{p_\rho^2}{m^2 V^2}} \simeq mV - \frac{1}{2} \frac{p_\rho^2}{mV},$$

so that eq.(18) can be eventually written in terms of an integration from 0 to ∞ :

$$\Psi(\rho, \zeta) = \mathcal{N} e^{2imV\zeta/\hbar} \int_0^\infty dp_\rho J_0\left(\frac{\rho p_\rho}{\hbar}\right) S(p_\rho) \exp \left[i \frac{p_\rho^2}{2\hbar mV} \zeta \right]. \quad (20)$$

Let us now examine various special cases of weight-functions $S(p_\rho)$ obeying the previous conditions: that is, well localized around a value $p_\rho \ll mV$.

^{||}For the sake of clarity, let us repeat that, when the phase-velocity V becomes (as in our case) the group-velocity, $V_g = V$, then the component p_ρ of \mathbf{p} acquires mV as its maximum value. It holds, moreover, $\sqrt{p^2 - p_\rho^2} = p_\parallel \equiv p_z$, which just equals p , since in the present case $V \equiv |\mathbf{V}| = V_z$.

3.1 Some examples of approximate Localized Solutions to the Schroedinger equation (*paraxial approximation*)

As already claimed, we are for the moment adopting the *paraxial approximation*, since it yields good, and interesting enough, results: Only in the subsequent Sections we shall go on to the exact, analytical approach.

First of all, let us consider the simple spectrum

$$S(p_\rho) = 4q p_\rho e^{-q p_\rho^2} \quad (21)$$

(with the dimensions, now, of the inverse of an Impulse), with

$$q \equiv \frac{\alpha}{m^2 V^2} \quad (22a)$$

so that the above conditions merely imply the dimensionless constant α to be

$$\alpha \gg 1 . \quad (22b)$$

In this case, also the total spectral-width Δp_ρ results to be $\Delta p_\rho \ll mV$: and this too supports the fact that our integral can indeed run till ∞ . In eq.(20), one can then perform (analytically) the integration, and get the solutions

$$\Psi(\rho, \zeta) \simeq \mathcal{N} 4q\hbar^2 e^{2imV\zeta/\hbar} \frac{1}{2Q} \exp \left[-\frac{\rho^2}{4\hbar(q\hbar - i\frac{1}{mV}\zeta)} \right] , \quad (23)$$

quantity q being still the one defined in eq.(22a), with $\alpha \gg 1$; while function Q is

$$Q \equiv \hbar(q\hbar - \frac{i}{2mV}\zeta) . \quad (24)$$

Equation (23) constitutes an interesting solution of the Schroedinger equation: It describes a wavepacket rigidly moving with the chosen speed V . The maximum of its intensity $|\Psi|^2$ occurs at

$$\rho = 0; \quad \zeta = 0 ,$$

and therefore also such a maximum travels with the speed V , as expected (since $\zeta = z - Vt$). For $\zeta = 0$ one gets [$\alpha \gg 1$]:

$$|\Psi(\rho, \zeta = 0)|^2 \simeq \mathcal{N}^2 4 \exp\left[-\frac{\rho^2}{2q\hbar^2}\right], \quad (25)$$

and the *transverse* localization $\Delta\rho$ of the wavepacket results to be

$$\Delta\rho = \frac{\hbar}{mV} \sqrt{2\alpha}, \quad (25')$$

which shows also the rôle of α (and therefore of q) in regulating the wavepacket (constant) transverse total width.

By contrast, putting $\rho = 0$ into eq.(23), we end up with the expression [still with $\alpha \gg 1$]:

$$|\Psi(\rho = 0, \zeta)|^2 \simeq \mathcal{N}^2 4 \frac{q^2 \hbar^2}{q^2 \hbar^2 + \frac{1}{4m^2 V^2} \zeta^2}, \quad (26)$$

which corresponds to

$$\Delta\zeta = \sqrt{e^2 - 1} \frac{2\alpha\hbar}{mV}.$$

Solution (26) is represented in Fig.2.

Let us briefly consider a few further possible spectra. We shall go on confining ourselves to the simple case of cylindrical symmetry, but analogous solutions can be *easily* found also for more general non-symmetrical cases.

As the second option, let us choose the new spectrum

$$S(p_\rho) = \frac{1}{p_\rho} e^{-qp_\rho^2}, \quad (27)$$

quantity q being defined in eq.(22a), and condition (22b) being enforced, so that $q \gg 1/(m^2 V^2)$ and, again, $\Delta p_\rho \ll mV$. Equation (20) yields the new solution

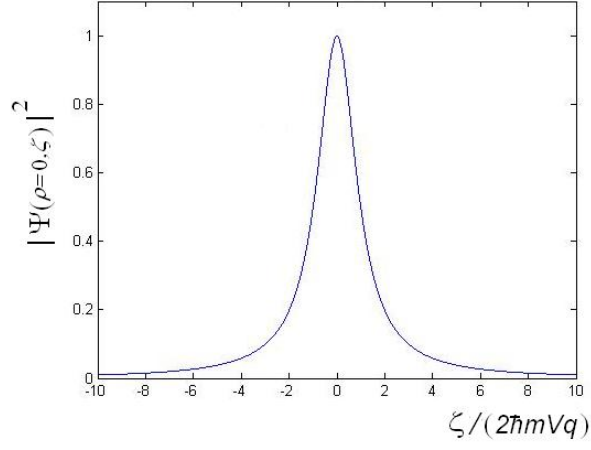


Figure 2: Behavior of $|\Psi(\rho = 0, \zeta)|^2$ in eq.(26), as a function of $\zeta/(2\hbar qmV)$.

$$\Psi(\rho, \zeta) \simeq \mathcal{N} \frac{1}{2} \gamma \left(0, \frac{\rho^2}{4Q} \right) \exp \left[\frac{i2mV}{\hbar} \zeta \right], \quad (28)$$

where function Q is defined in eq.(24), and γ , here, is the “incomplete gamma function”. [22]

$$\gamma(0, \mathcal{A}) = -\gamma(-1, \mathcal{A}) - \mathcal{A}^{-1} e^{-\mathcal{A}}$$

with

$$\begin{aligned} \gamma(-1, \mathcal{A}) &\equiv -\mathcal{A}^{-1} e^{-\mathcal{A}} \Phi(1, 0; \mathcal{A}) \\ &\equiv -\mathcal{A}^{-1} e^{-\mathcal{A}} [1 - \Phi(1, 0; \mathcal{A})], \end{aligned}$$

function Φ being the “Probability Integral”, that in the present case can be defined as

$$\Phi(1, 0; \mathcal{A}) \equiv \frac{1}{\Gamma(1)} \int_0^\infty dx \frac{\alpha - e^{-\mathcal{A}x}}{1 - e^{-x}}.$$

The maximum, also for solution (27), occurs at $\rho = \zeta = 0$.

As a third option, we choose

$$S(p_\rho) = qp_\rho e^{-qp_\rho^2} I_0\left(\frac{sp_\rho}{\hbar}\right) \quad (29)$$

always with $\alpha \gg 1$, quantity q being given by eq.(22a), s a constant with the dimensions of a Length (regulating the spectrum bandwidth), and I_0 being the Modified Bessel Function; one gets from eq.(20) the further new solution

$$\Psi(\rho, \zeta) \simeq \mathcal{N} \frac{q\hbar}{2Q} e^{\frac{i2mV}{\hbar}\zeta} \exp\left[\frac{s^2 - \rho^2}{4Q}\right] J_0\left(\frac{s\rho}{2Q}\right) . \quad (30)$$

As the last option, let us choose

$$S(p_\rho) = qp_\rho e^{-qp_\rho^2} J_0(sp_\rho) , \quad (31)$$

from eq.(20) it follows the fourth solution

$$\Psi(\rho, \zeta) \simeq \mathcal{N} \frac{q}{2Q} e^{\frac{i2mV}{\hbar}\zeta} \exp\left[-\frac{s^2 + \rho^2}{4Q}\right] I_0\left(\frac{s\rho}{2Q}\right) . \quad (32)$$

4 Exact Localized Solutions to the Schroedinger equation (for arbitrary frequency spectra)

Our aim is now to construct new analytical solutions to the Schroedinger equation, by following an exact (not approximate) approach. Let us, then, go back to eq.(1), and to its Bessel-beam solution (11), where, as before, relation (12b) holds: $p_\rho = \sqrt{2mE - p_z^2}$, with $E = \omega\hbar$.

The *condition* for obtaining a Localized Solution (cf. Fig.3) is that

$$E = Vp_z + b , \quad (33a)$$

with b a positive constant (bearing the dimensions of an Energy, and regulating the position of the chosen straight-line in the plane (E, p_z)); which corresponds in particular, on using eq.(12b), to the adoption of the integration limits

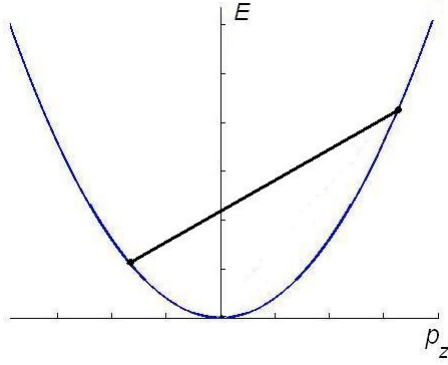


Figure 3: This time, the parabola and the chosen straight-line have equations $E = p_z^2/(2m)$ and $E = Vp_z + b$, respectively. The intersections of this straight-line with the parabola are now two, whose corresponding values are given in eq.(33b). Inside the parabola $p_\rho^2 \geq 0$.

$$E_{\pm} = mV^2 \left(1 \pm \sqrt{1 + \frac{2b}{mV^2}} \right) + b. \quad (33b)$$

Localized Solutions can therefore be obtained by the following superpositions (integrations over the frequency, or the energy) of Bessel-beam solutions:

$$\Psi(\rho, z, \zeta) = e^{\frac{-ib}{\hbar V} z} \int_{E_-}^{E_+} dE J_0(\rho p_\rho / \hbar) S(E) e^{i \frac{E}{\hbar V} \zeta}, \quad (34)$$

together with

$$p_\rho = \frac{1}{V} \sqrt{-E^2 + (2mV^2 + 2b)E - b^2}. \quad (35)$$

Notice that the in eq.(34) [as well as in eq.(39) below], the solution Ψ depends on z , besides via ζ , only via a phase factor; the modulus $|\Psi|$ of Ψ goes on depending on z (and on t) only through the variable $\zeta \equiv z - Vt$.

4.1 Particular exact Localized Solutions

We want now to re-write the integral \mathcal{I} appearing in the r.h.s. of eq.(34) so that its integration limits are -1 and $+1$, respectively; that is, in the form

$$\mathcal{I} = \int_{-1}^1 du S(u) J_0\left(\frac{\rho\sqrt{P}}{\hbar}\sqrt{1-u^2}\right) e^{i f(\zeta)u} ,$$

quantity $f(\zeta)$ being an arbitrary dimensionless function. To obtain this, we have to *look for* a transformation of variables [with A and B constants, with the dimensions of an Energy, to be determined]

$$E = Au + B \tag{36}$$

such that

$$p_\rho^2 = P(1-u^2) ; \quad u_+ = 1 ; \quad u_- = -1 , \tag{36'}$$

P being a suitable constant (with the dimensions of an Impulse square). On writing $V^2 p_\rho^2 = E(\hbar V^2 M - E) - b^2$, with $\hbar M \equiv 2m + 2b/V^2$, after some algebra one finds that it must be

$$A = \sqrt{P} V ; \quad B = mV^2 + b ; \quad P = m^2 V^2 + 2mb . \tag{37}$$

Indeed, one can verify (by some more algebra) that eqs.(36)-(37) imply, as desired, that $u_- = -1$ and $u_+ = 1$.

In conclusion, the transformation

$$E = mV^2 \sqrt{1 + \frac{2b}{mV^2}} u + mV^2 + b \tag{38}$$

does actually allow writing solution (34) in the form [recall that $E = Au + B \implies dE = Adu$]

$$\Psi(\rho, \eta, \zeta) = \mathcal{N} A e^{\frac{imV}{\hbar}\eta} \int_{-1}^1 du S(u) J_0\left(\frac{\rho}{\hbar}\sqrt{P}\sqrt{1-u^2}\right) e^{\frac{iA\zeta}{\hbar V}u} , \tag{39}$$

with

$$\eta \equiv z - vt ,$$

where $v \equiv V + b/(mV)$. Equation (39) is exactly, analytically integrable when S is a constant or a suitable exponential.

Let us choose the complex exponential function (which will easily enter as an element in a Fourier expansion)

$$\bar{S}(E) = a_n e^{\frac{2\pi i}{D} n E}, \quad (40)$$

with n an integer, and $D \equiv E_+ - E_- = 2mV^2 \sqrt{1 + 2b/(mV^2)}$, while a_n are constant quantities (with dimensions of the inverse of an Energy). On remembering that $E = Au + B$, such a spectrum can be written in terms of u as

$$\bar{S}(u) = a_n e^{i\pi n u} e^{i\frac{2\pi}{D} n B} \quad (40')$$

(still with the dimensions of an inverse Energy). After some more algebra, the analytic exact solution to the Schroedinger equation, corresponding to spectrum (41), results to be[11]

$$\Psi(\rho, \eta, \zeta) = \mathcal{N} a_n 2A \frac{\sin Z}{Z} e^{\frac{imV}{\hbar} \eta} e^{i\frac{2\pi}{D} n B}, \quad (41)$$

where A , B , P are given by eqs.(37) and

$$Z \equiv \sqrt{\left(\frac{A}{\hbar V} \zeta + n\pi\right)^2 + \frac{P}{\hbar^2} \rho^2}. \quad (42)$$

Equation (41), as we have just seen, is a particular exact Localized Solution to the Schroedinger equation; but we are going to utilize it essentially as *an element* of suitable superpositions. Before going on, however, we wish to depict in Figs.4 an elementary solution: namely, the square magnitude of the simple solution corresponding, in eq.(34), to the *real* exponential

$$S(E) = s_0 \exp[a(E - E_+)], \quad (43)$$

a being a positive number, endowed with the dimensions of an inverse Energy, as well as s_0 . When $a = 0$, one ends up with a solutions similar to Mckinnon's[23]. Spectrum (43) is exponentially concentrated in the proximity of E_+ , where it reaches its maximum value; and becomes more and more concentrated (on the left of E_+ , of course) as the arbitrarily

chosen value of a increases. To perform the integration in eq.(34), it is once more useful to operate the variable transformation (36) and go on to eq.(39), spectrum (43) assuming now the form

$$S(u) = s_0 e^{-aE_+} e^{aB} e^{aAu} .$$

Performing the integration in eq.(39), by a process similar to the one which led us to eq.(41), in the present case we get

$$\Psi(\rho, \eta, \zeta) = \mathcal{N} s_0 2V \sqrt{P} \exp[i \frac{mV}{\hbar} \eta] \exp[-aV \sqrt{P}] \frac{\sin Y}{Y} \quad (44a)$$

where

$$Y \equiv \frac{\sqrt{P}}{\hbar} \sqrt{\rho^2 - (\hbar a V + i\zeta)^2} , \quad (44b)$$

quantity P having been defined in eq.(37); and one should remember that $\eta \equiv z - vt$ is a function of b .

Equations (44) appear to be the simplest closed-form solutions (see Figs.4) to the Schroedinger equation, since they do not need any recourse to series expansions of the type exploited in the following Subsection. However, the solutions that we shall construct below can correspond to spectra more general than (43); for instance, to the gaussian spectrum, which possesses two advantage w.r.t. spectrum (43): it can be easily centered around any value of u , that is, around any value \bar{E} of E in the interval $[E_-, E_+]$, and, when increasing its concentration in the surrounding of \bar{E} , its “spot” transverse width does not increase, at variance with what happens for spectrum (43). Anyway, the exact solutions (44) are noticeable, since they are really the simplest ones.

Some physical (interesting) comments on the results in eqs.(44) and Figs.4 will appear elsewhere. Here, let us add only a few further Figures and some brief comments. Let us first recall that, as predicted in the first one of Refs.[6], the Localized (Nondiffracting) Solutions to the ordinary wave equations resulted to be roughly *ball-like* when their peak-velocity is subluminal[11], and *X-shaped*[4,7] when superluminal.

Now, normalizing ρ and ζ , we can write eq.(44b) as

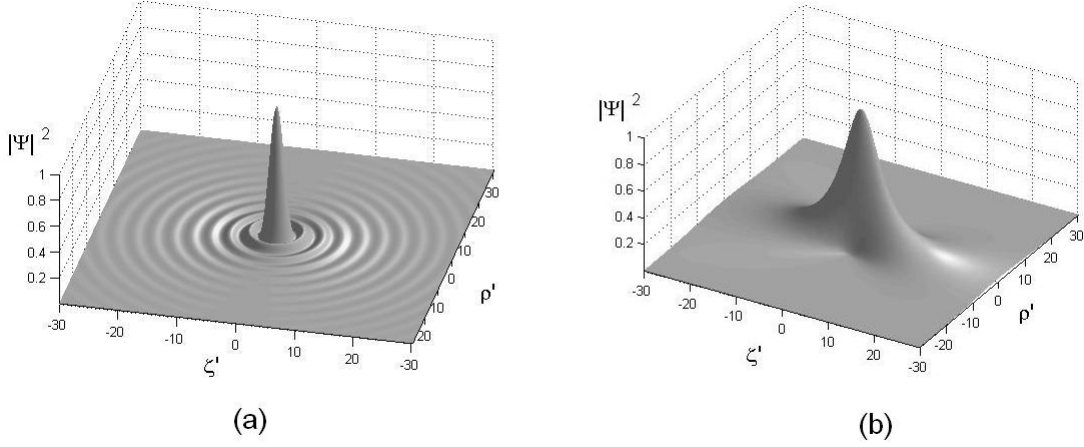


Figure 4: In these figures we depict an elementary solution: namely, the square magnitude of the simplified solution, eq.(44a), corresponding to the *real* spectrum $S(u) = s_0 \exp[(E - E_+)a]$, as a function of $\rho' \equiv \rho\sqrt{P}/\hbar$ and of $\zeta' \equiv \zeta\sqrt{P}/\hbar$. Quantity a is a positive number [when $a = 0$ one ends up with a solutions similar to Mckinnon's[23]], while b for simplicity has been chosen equal to zero. Figure (a) corresponds to $a = E_+/5$, while figure (b) corresponds to $a = 5E_+$. For the properties of the spectral function (43), see the text.

$$Y = \sqrt{\rho'^2 - (\bar{A} + i\zeta')^2}$$

with $\rho' \equiv \sqrt{P}\rho/\hbar$ and $\zeta' \equiv \sqrt{P}\zeta/\hbar$, quantity P being given by the last one of eqs.(37), namely $P = m^2V^2 + 2mb$, while $\bar{A} \equiv aA = \sqrt{P}aV$. For simplicity, let us confine ourselves to the case $b = 0$, forgetting now about the more interesting cases with $b \neq 0$; therefore, it will hold the simple relation

$$\bar{A} = maV^2.$$

In the present case of the Schroedinger equation, we can observe the following.

If we choose $\bar{A} = 0$, which can be associated with $V = 0$, we get the solutions in Figs.5: that is, a ball-like structure.

By contrast, if we increase the value of \bar{A} , by choosing e.g. $\bar{A} = 20$ (which can be associated with larger speeds), one notices that also a X-shaped structure starts to

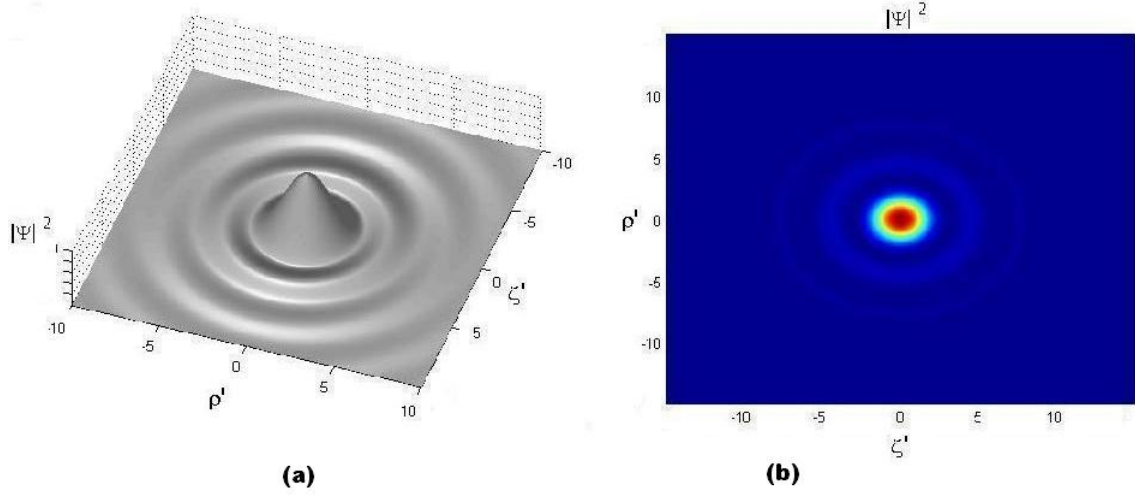


Figure 5: In these, and the following Figures 6, 7 and 8, we depict the square magnitude of some more solutions of the type (44a), normalized with respect to ρ and ζ ; still assuming for simplicity $b = 0$, so that $\bar{A} = maV^2$. The present figures show the “ball-like” structure that one gets, as expected, when $\bar{A} = 0$ (see the text, also for the definitions of ρ' and ζ'). Fig.(b) shows the projection on the plane (ρ', ζ') of the 3D plot shown in Fig.(a).

contribute: See, e.g., Fig.6.

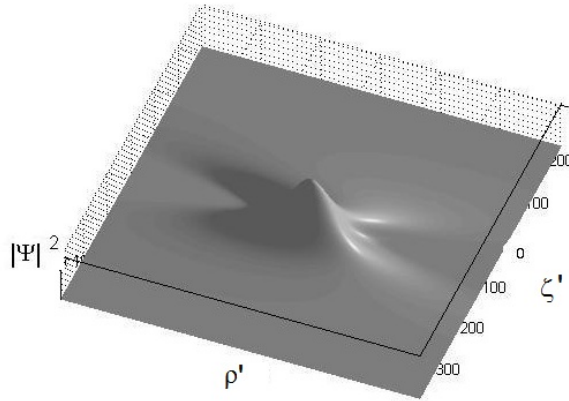


Figure 6: The solution, under all the previous conditions, with an increased value of \bar{A} , namely with $\bar{A} = 20$. An X-shaped structure starts to appear, contributing to the general form of the solution (see the text).

To have a preliminary idea of the “internal structure” of our soliton-like solutions to the (ordinary) Schroedinger equation, let us plot, instead of the square magnitude of Ψ , its real or imaginary part: Let us choose its real part, or rather the square of its real part. Then even in the $\bar{A} = 0$ case one starts to see the appearance of the X shape, which becomes more and more evident as the value of \bar{A} increases: In Figs.7 we show the projections on the plane (ζ', ρ') of the real-part square for the solutions with $\bar{A} = 5$ and $\bar{A} = 50$, respectively. Further attention to such aspects will be paid elsewhere.

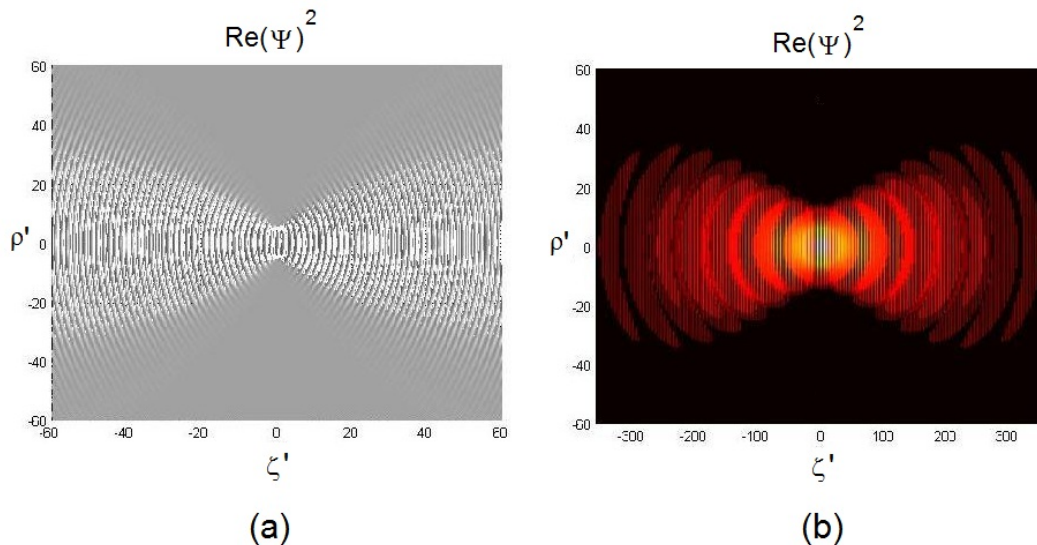


Figure 7: To get a preliminary idea of the “internal structure” of our soliton-like solutions, it is useful to have recourse (see the text) to the real part of Ψ . In these Figures we plot the projections on the plane (ζ', ρ') of the real-part square for the solutions with $\bar{A} = 5$ (figure (a)) and $\bar{A} = 50$ (figure (b)), respectively.

But the (square of the) real part of Ψ does show, in 3D, also some “internal oscillations”: Cf., e.g., Fig.8 corresponding to the value $\bar{A} = 5$. We shall face elsewhere, however, topics like their possible connections with the de Broglie picture of quantum particles, et alia.

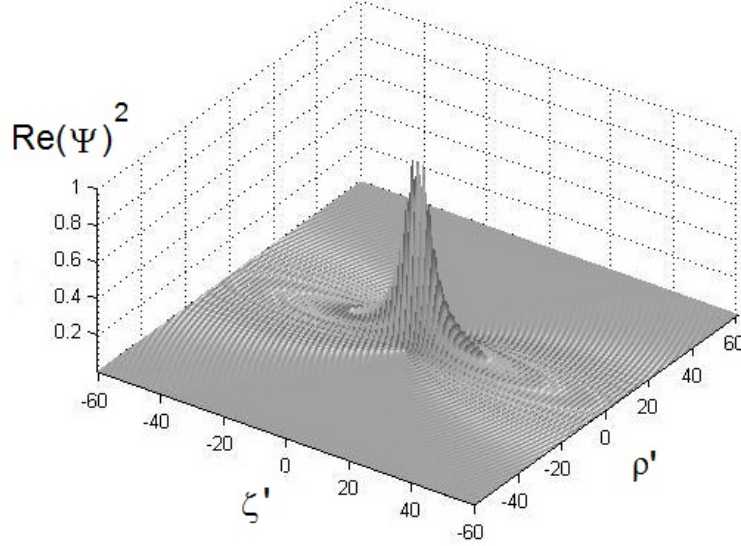


Figure 8: The (square of the) real part of Ψ shows, in 3D, also some “internal oscillations”: this Figure corresponds, e.g., to the value $\bar{A} = 5$.

4.2 A general exact Localized Solution

Let us go back to our spectrum $S(E)$ in eq.(40). Since in our fundamental equation (34) the integration interval is limited [$E_- < E < E_+$], in such an interval *any* spectral function $S(E)$ whatever can be expanded into the Fourier series

$$S(E) = \sum_{n=-\infty}^{\infty} a_n e^{i\frac{2\pi}{D}nE}, \quad (45)$$

with

$$a_n = \frac{1}{D} \int_{E_-}^{E_+} dE S(E) e^{-i\frac{2\pi}{D}nE}, \quad (46)$$

quantity $S(E)$ being an *arbitrary* function, and D being still defined as $D \equiv E_+ - E_-$.

Inserting eq.(45) into eq.(34), and following the same procedure exploited in the previous Subsection (in particular, going on again from E to the new variable u), we end

up —after normalization— with the *general exact localized solution* to the Schroedinger equation:

$$\Psi(\rho, \eta, \zeta) = \mathcal{N} 2A e^{i\frac{mV}{\hbar}\eta} \sum_{n=-\infty}^{\infty} a_n \exp\left[i\frac{2\pi}{D}nB\right] \frac{\sin Z}{Z}, \quad (47)$$

where Z is defined in eq.(42), and the coefficients a_n are given by eq.(46).

It is worthwhile to note that, even when truncating the series in eq.(47) at a certain value $n = N$, the solutions obtained is *still* an exact LS of the Schroedinger equation!

5 About finite-energy Localized Solutions to the Schroedinger equation

The solutions found above, even if very instructive, are ideal solutions which are not square integrable; and cannot be accepted in QM. It is important, therefore, to show how to construct finite-energy solutions.

Let us obtain localized solution to the Schroedinger equation endowed with *finite energy*, by starting from eqs.(44). First of all, one has to integrate over b by adopting a spectrum $S(b)$ strongly bumped around a value b_0 : We already know, indeed, that spectra of this type are required in order to get solutions that are non-diffracting all along a certain field-depth.

Then, it can be easily seen that the finite-energy solution, Ψ_{fe} , can be preliminarily written as

$$\Psi_{\text{fe}} = \mathcal{N} \frac{s_0 V \sqrt{P}}{iY} (I_- - I_+), \quad (48)$$

where I_- and I_+ are two (dimensionless) integrations over b from 0 to infinity (quantity b having been defined in eq.(33a), and therefore having the dimensions of an Energy), while s_0 appears in eq.(43).

Let us now pass from b , defined in eq.(33a), to the new variable $w \equiv \sqrt{P}$. One has to choose a spectrum $S(w)$ corresponding to a $S(b)$ concentrated around a specific value of b ; let us therefore adopt the gaussian function

$$\mathcal{S}(w) = \frac{m\sqrt{q}}{\sqrt{\pi}\hbar w} \exp[-q(w - w_0)^2], \quad (49)$$

with $w_0 > mV > 0$.

When we go on from b to the new variable $w \equiv \sqrt{P}$ (where P depends on b), the two quantities I_- and I_+ become integrations over w from mV to ∞ . After further calculations, and using relation 3.322.1 in ref.[22], one obtains that

$$I_{\pm} = \frac{\sqrt{q}}{U} e^{-qw_0} e^{\frac{imV}{2\hbar}z} \exp\left[\frac{W_{\pm}^2}{U^2}\right] \left[1 - \Phi\left(\frac{W_{\pm}}{U} + \frac{mV}{2}U\right)\right], \quad (50)$$

where

$$U \equiv 2\sqrt{q + \frac{i\hbar}{2m}t}; \quad W_{\pm} \equiv -2qw_0 + aV \pm i\frac{Y}{\sqrt{P}},$$

quantity Y having been defined in eq.(44b).

We have therefore shown that realistic (finite-energy) Localized Solutions exist also to the Schroedinger equation; they will be non-diffracting only till a certain finite distance (depth of field). The analysis of explicit, particular examples will be presented elsewhere.

6 The case of non-free particles

Let us consider now the case of a particle in the presence of a potential: for simplicity, let us confine ourselves to the case of a *cylindrical potential*.

Namely, let us consider the Schroedinger equation with a potential of the type $U(\rho)$:

$$-\frac{\hbar^2}{2m} \left(\nabla_{\perp}^2 + \frac{\partial^2}{\partial z^2} \right) \psi + U(\rho)\psi - i\hbar \frac{\partial \psi}{\partial t} = 0 \quad (51)$$

Now, we can use the method of separation of variables writing $\psi = R(x, y)Z(z)T(t)$. With this, we get the well known solutions

$$T = e^{-\frac{i}{\hbar} E t} \quad (52)$$

$$Z = e^{ip_z z/\hbar} \quad (53)$$

and the eigenvalue equation

$$-\hbar^2 \nabla_{\perp}^2 R + 2m U(\rho) R = \Lambda^2 R \quad (54)$$

with

$$\Lambda^2 = 2mE - p_z^2 \quad (55)$$

Supposing a potential $U(\rho)$ that only allows transverse bound states (as the parabolic potential), we will find eigenfunctions $R_n(x, y)$ and discrete (degenerate) eigenvalues Λ_n^2 .

We can construct more general solutions

$$\Psi = \sum_n f_n R_n(x, y) e^{ik_z z/\hbar} e^{-\frac{i}{\hbar} E t} \quad (56)$$

with

$$2mE = p_z^2 + \Lambda_n^2 \quad (57)$$

Considering $p_z \geq 0$ (forward propagation), the constraint (57) defines a set of parabolas (something like the modes in a waveguide: Cf. Refs.16). Chosen a certain Λ_n^2 , once a value for p_z is given, the value of E gets fixed.

To obtain from (56) a train of localized pulses, i.e., a wavefunction $\Psi(x, y, z - Vt)$, we must have

$$E = V p_z \quad (58)$$

So, from conditions (57) and (58), p_z must assume the values

$$p_z = mV \left(1 \pm \sqrt{1 - \frac{1}{m^2 V^2} \Lambda_n^2} \right) \quad (59)$$

with

$$\Lambda_n \leq mV \quad (60)$$

Figure 9 illustrates the situation. The values to E and p_z that furnish localized pulse trains are given by the intersection between the parabolas defined by eq.(57) and the straight line defined by eqs.(58). Note that in these cases the series (56) will be always truncated (finite number of terms), due the condition (60). We also have to note that, for any given λ_n^2 , one gets two possible values of k_z (see eq.(59)), as it can be observed from Fig.9, in which the straight line cuts each parabola twice.

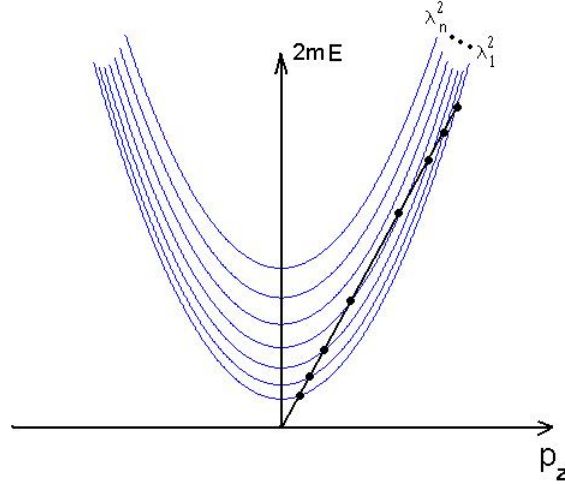


Figure 9: In the case of a particle in the presence of a cylindrical potential, the values to E and p_z that furnish Localized Pulse trains are given by the intersection between the parabolas in eq.(57) and the straight line in eq.(58): see the text. It can be noticed that, for any given λ_n^2 , one gets two possible values of k_z (cf. eq.(59)), since the straight line cuts each parabola twice. See the text, and cf. also Refs.[17].

For our purpose, the superposition has to be

$$\Psi(x, y, z - Vt) = \sum_n f_n R_n(x, y) e^{ip_{zn}(z - Vt)/\hbar} \quad (61)$$

with

$$p_z = mV \left(1 \pm \sqrt{1 - \frac{1}{m^2 V^2} \Lambda_n^2} \right) \quad (62)$$

and

$$\Lambda_n \leq mV \quad (63)$$

In principle, any set of coefficients f_n will furnish *trains of localized waves*.

Observation1: If we look for a square-integrable wave function, we can start from superposition (56) and integrate its terms over p_z around each p_{zn} , respectively (as we already did in our papers on X-type pulses propagating along wave-guides[17]). But in the present case, in general, the group-velocities defined at the points p_{zn} will *not* be the same, as it happened in the waveguide case; and we will therefore meet a kind of intermodal dispersion, besides the group-velocity dispersion. Let us recall, incidentally, that such an intermodal dispersion did not occur in the case of X-type waves, traveling in metallic wave-guides, due the peculiar fact that the group-velocities defined at those points were always the same). After the integration, we can obtain an envelope with a train of pulses (or just one pulse) inside it. The envelope will suffer dispersion, but the train of pulses inside it will not.

More general localized wave trains can be obtained using the relation $E = Vp_z + b$, with b a positive constant.

In the case of potentials like $U(\rho)$, one can search for solutions with cylindrical symmetry, for simplicity. However, solutions without this symmetry can be investigated: and they will be interesting for an analysis of angular momentum.

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